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FINITE ELEMENT ANALYSIS OF CONVECTIVE HEAT TRANSFER IN POROUS MEDIA

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SUMMARY

The finite element method is used to analyse convective heat transfer in a porous medium. Convection past a vertical surface embedded in the medium and convection in a confined porous medium enclosure are analysed using the above method. The results are compared with those available in the literature and the agreement is found to be good. The method is applicable for two-dimensional analysis in a porous body of any arbitrary shape. The restriction of the boundary layer assumption is relaxed.

KEY WORDS Porous media Convective heat transfer Finite element analysis.

1. INTRODUCTION

Convective heat transfer in a fluid-saturated porous medium occurs in many geophysical and engineering systems. The study of such heat transfer has important applications in fields like petroleum engineering, geothermal systems, nuclear engineering and insulation technology. The subject has hence received much attention in the past two decades.

In a review article, Cheng[1] lists the various cases of heat transfer by free and mixed convection from flat plates into a porous medium. The study of heat transfer in porous medium enclosures of various shapes has also received much attention.[2–6]

An analysis for steady free convection about a vertical flat plate embedded in a porous medium with wall temperature varying as a power function of distance from the origin was first made by Cheng and Minkowicz.[7] Later, Johnson and Cheng[8] made a systematic analysis considering the possibility of similarity solutions for various wall temperature functions. It was found in Reference 8 that similarity solutions can be obtained for wall temperature varying as a power function or an exponential function. Both the above analyses suffer from the disadvantage that each type of temperature function has to be solved separately. Solutions have been obtained for the mixed convection problem with wall temperature as an arbitrary function of distance in the form of perturbations to the isothermal case.[9] The present method using finite elements allows arbitrary variation of temperature along the wall. Also, no boundary layer approximations need be made as was done in the earlier cases. The method can easily be extended to study other configurations of porous media. Here we analyse the case of a porous medium rectangular enclosure with the vertical walls at different temperatures and the case of a heated vertical plate in a porous medium.

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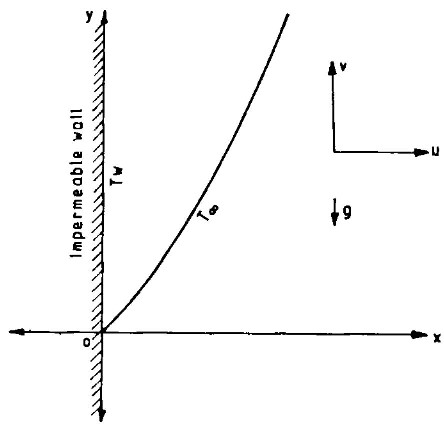
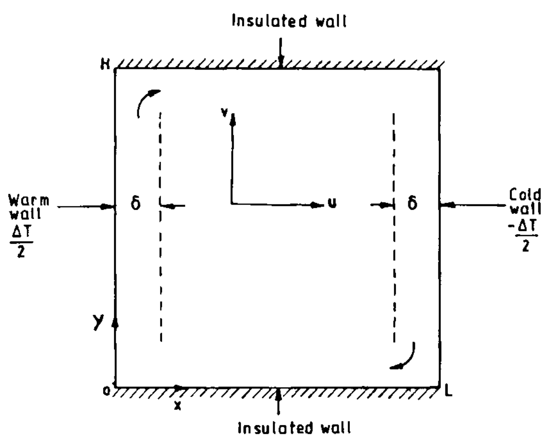
2. ANALYSIS

Consider the problem of free convection in a saturated porous medium adjacent to a non-isothermal vertical impermeable surface. The physical situation is shown in Figure 1(a), where x and y are Cartesian co-ordinates in the horizontal and vertical directions respectively, with the positive x-axis pointing towards the porous medium. The origin of the co-ordinate system is chosen at that point of the impermeable surface where the wall temperature begins to deviate from that of the surrounding fluid. The wall temperature is assumed to vary along y from the origin arbitrarily with $T_w > T_\infty$.



(a) Vertical non-isothermal surface embeded in the porous medium



(b) Rectangular porous medium enclosure

Figure 1. Physical model and co-ordinate system

We assume that:

(i) The convective fluid and the porous medium are everywhere in local thermodynamic equilibrium.
(ii) There is no phase change of the fluid in the medium.
(iii) The properties of the fluid and of the porous medium are homogeneous and isotropic.
(iv) The Boussinesq approximation can be applied.

Under these assumptions the governing equations are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u=-\frac{K}{\mu}\frac{\partial P}{\partial x}, \tag{2}$$

$$v=-\frac{K}{\mu}\left(\frac{\partial P}{\partial y}+\rho g\right), \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \tag{4}$$

$$\rho=\rho_\infty[1-\beta(T-T_\infty)], \tag{5}$$

where u and v are Darcy's velocities in the x- and y-direction respectively; ρ, μ and β are the density, viscosity and thermal expansion coefficient respectively of the fluid; K is the permeability of the porous medium; $\alpha = k_m/(\rho_\infty c)_F$ is the equivalent thermal diffusivity, with $(\rho_\infty c)_F$ denoting the product of density and specific heat of the convecting fluid, and k_m the thermal conductivity of the saturated porous medium given by $k_m=(1-\varepsilon)k_S+\varepsilon k_F$, where ε is the porosity of the medium and k_S and k_F are the thermal conductivities of the solid and convective fluid respectively; T, P and g are temperature, pressure and gravitational acceleration respectively. The subscript '∞' refers to conditions far away from the heated surface. The boundary conditions for the problem are

$$\text{at } x=0: \quad u=0, \quad T=T_w(y); \tag{6}$$

$$\text{as } x\to\infty: \quad v\to 0, \quad T=T_\infty. \tag{7}$$

The continuity equation (1) can be satisfied automatically by introducing the streamfunction ψ as

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\frac{\partial\psi}{\partial x}. \tag{8}$$

Eliminating P from equations (2) and (3) by cross-differentiating and by making use of (5), the resulting equation in terms of ψ is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\frac{Kg\beta}{\nu}\frac{\partial T}{\partial x}. \tag{9}$$

In terms of ψ, the energy equation (4) can be written as

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}\frac{\partial^2 T}{\partial y^2}\right). \tag{10}$$

Equations (9) and (10) constitute the governing equations for free convection in a porous medium.

The finite element method is used to solve these equations. Each equation has both the variables ψ and T—we thus have two coupled equations which have to be solved simultaneously.

The simplex three-noded triangular element is used for the analysis. The variations of T and ψ inside the element are given by

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 = [N]\{T\}, \tag{11}$$

$$\psi = N_1 \psi_1 + N_2 \psi_2 + N_3 \psi_3 = [N]\{\psi\}, \tag{12}$$

where N_1, N_2, N_3 are the shape functions[6] given by

$$N_i = \frac{a_i + b_i x + c_i y}{2A}, \quad i = 1, 2, 3. \tag{13}$$

We use the popular Galerkin method for the element formulation. For equation (10) this becomes

$$\int_V N^{\mathrm{T}} \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{1}{\alpha} \left(\frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} \right) \right] \mathrm{d}V = 0. \tag{14}$$

The integral when evaluated gives an equation of the form

$$[K_1]\{T\} = \{F_1\}. \tag{15}$$

$[K_1]$ itself is a function of ψ. Thus the equation contains both ψ and T. For the three-noded triangular element,

$$\begin{aligned}
[K_1] = {} & \frac{1}{12A\alpha} \begin{bmatrix} c_1\psi_1 + c_2\psi_2 + c_3\psi_3 \\ c_1\psi_1 + c_2\psi_2 + c_3\psi_3 \\ c_1\psi_1 + c_2\psi_2 + c_3\psi_3 \end{bmatrix} [b_1 \;\; b_2 \;\; b_3] \\
& - \frac{1}{12A\alpha} \begin{bmatrix} b_1\psi_1 + b_2\psi_2 + b_3\psi_3 \\ b_1\psi_1 + b_2\psi_2 + b_3\psi_3 \\ b_1\psi_1 + b_2\psi_2 + b_3\psi_3 \end{bmatrix} [c_1 \;\; c_2 \;\; c_3] \\
& + \frac{1}{4A} \left\{ \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^2 & c_2 c_3 \\ c_1 c_3 & c_2 c_3 & c_3^2 \end{bmatrix} \right\}.
\end{aligned} \tag{16}$$

F_1 is zero in the present case (since there are no convection or heat flux boundary conditions).

Similarly, applying Galerkin's criterion to equation (9), we have

$$\int_V N^{\mathrm{T}} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{Kg\beta}{\nu} \frac{\partial T}{\partial x} \right) \mathrm{d}V = 0. \tag{17}$$

This yields the equation

$$[K_2]\{\psi\} = \{F_2\}, \tag{18}$$

where

$$[F_2]=\frac{Kg\beta}{6v}\begin{Bmatrix} b_1T_1+b_2T_2+b_3T_3 \\ b_1T_1+b_2T_2+b_3T_3 \\ b_1T_1+b_2T_2+b_3T_3 \end{Bmatrix}, \tag{19}$$

$$[K_2]=\frac{1}{4A}\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix}+\frac{1}{4A}\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}. \tag{20}$$

It is to be noted that the stiffness matrix $[K_1]$ is asymmetric.

The element matrices given by (15) are assembled to get the global matrix equation, which is solved for $\{T\}$ by the Gaussian elimination technique. Initially, the value of $\{\psi\}$ is taken to be zero for the first iteration. The values of $\{T\}$ obtained are then used to solve the global matrix equation obtained by assembly of the matrices given by (18) for $\{\psi\}$. These $\{\psi\}$-values are now used for the next iteration in the calculation of $\{T\}$. The two equations are thus solved simultaneously by iteration. For the solutions to converge it is necessary that the mesh be fine. The values of $\{\psi\}$ at the wall are forced to be zero (since $u=0$ at $x=0$). The temperatures at the wall nodes are also incorporated.

3. RESULTS AND DISCUSSION

Vertical heated plate embedded in a porous medium

For purposes of comparison we have taken up the case of free convection about a dike.[7] When the hot intruded magma is in contact with the cooler subsurface environment, its outer surface will be chilled to form a thin glass selvage. The interior of the intrusive will continue to solidify when heat is transferred from the intrusive (dike) to the surroundings. When hot intrusive is trapped in an aquifer, however, its cooling rate is governed by heat convection. Thus it has been speculated that hot dike complexes in a volcanic region can provide an energy source for heating of the ground-water.[7] The surface temperature of the dike is assumed to be 200°C. The length of the dike is assumed to be 300 m and the dike is intruded in an aquifer at 15°C. The physical properties used[7] for the computations are $\beta=1{\cdot}8\times10^{-4}\,^{\circ}\mathrm{C}^{-1}\,\mathrm{m}^{-1}$, $\rho_\infty=10^6\,\mathrm{g\,m^{-3}}$, $c=1\,\mathrm{cal\,g^{-1}\,^{\circ}C^{-1}}$, $\mu=0{\cdot}27\,\mathrm{g\,s^{-1}\,m^{-1}}$, $k_\mathrm{m}=0{\cdot}58\,\mathrm{cal\,s^{-1}\,^{\circ}C^{-1}\,m^{-1}}$ and $K=10^{-12}\,\mathrm{m^2}$. The values of β, ρ_∞, c and μ are the data for ground-water evaluated at the mean film temperature.

A region of area 300 m by 140 m adjacent to the dike has been discretized and a finite element mesh of 256 nodes and 450 elements used. The finite element formulation described is utilized to obtain the temperature and streamfunction values at the nodes by the procedure outlined earlier. This output is plotted in the form of isotherms in the region (Figure 2).

The figure shows the boundary layer thickness increasing from zero at the origin to 82 m at a height of 300 m. Cheng and Minkowicz get the corresponding boundary layer thickness at this height as 80 m.[7]

Although the present model takes into account the conductive heat transfer in the vertical direction and makes no boundary layer approximations, the heat transfer results vary very little from those of Reference 7. This only confirms that it is quite permissible to make boundary layer approximations for such problems.

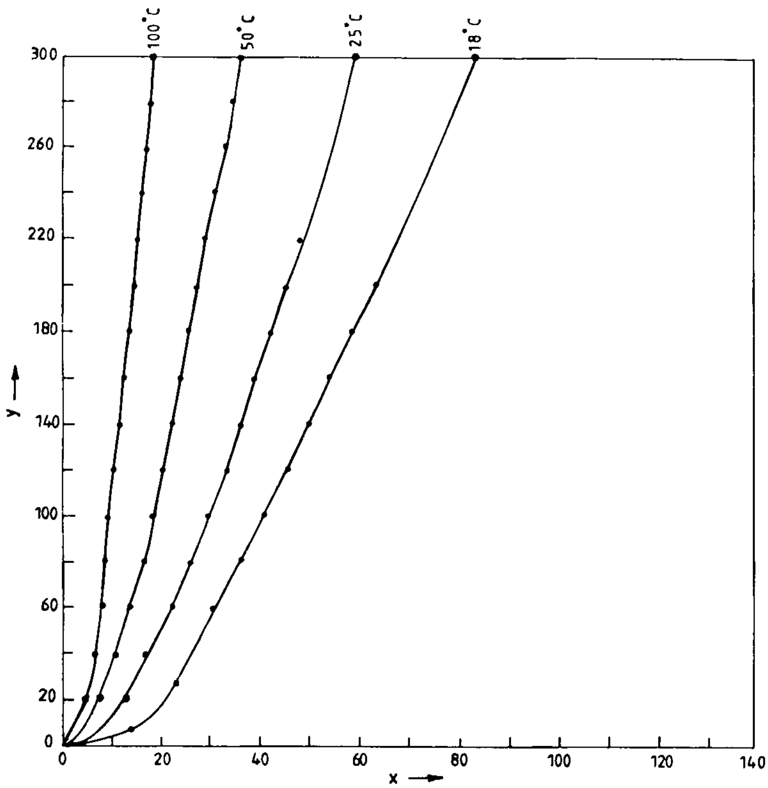


Figure 2. Isotherms for a dike with uniform wall temperatures

From the temperature distribution obtained, the standard dimensionless variables Nu and Ra were calculated. In the porous medium these are defined as

$$Ra_y = \rho_\infty g \beta K (T_w - T_\infty) y / \mu\alpha,$$

$$Nu_y = \frac{h_y}{k_m} = \frac{q_x y}{(T_w - T_\infty) k_m},$$

where h is the local heat transfer coefficient and q_x is the local heat transfer rate given by $-k_m \, \partial T / \partial x$.

For the dike we have considered, a total heat transfer rate of $8{\cdot}1 \times 10^6$ cal h^{-1} m^{-2} and an average heat transfer coefficient of 148·5 cal h^{-1} m^{-2} °C^{-1} are obtained. The corresponding values obtained by Cheng and Minkowicz are $8{\cdot}3 \times 10^6$ cal h^{-1} m^{-2} and 150 cal h^{-1} m^{-2} °C^{-1}.[7]

Next, a temperature distribution as a power function of distance along the wall given by

$$T_w(y) = T_\infty + A y^\lambda$$

is assumed. Solutions have been obtained for different values of λ (−0·33, −0·25, 0, 0·25, 0·5, 0·75 and 1) and the corresponding values of Nu and Ra are calculated in each case.

A plot of $Nu_y / Ra_y^{1/2}$ against λ is shown in Figure 3. These values are obtained at the location $y = 300$ m. The similarity solution of Cheng and Minkowicz yields a constant value of $Nu/Ra^{1/2}$

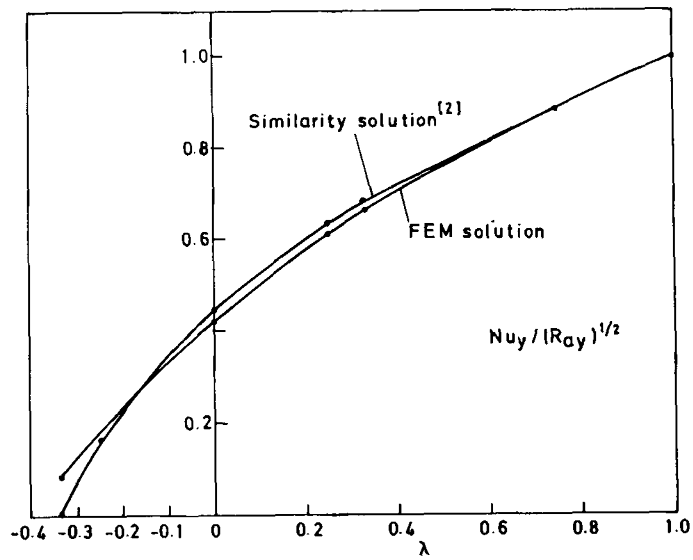


Figure 3. Comparison of heat transfer results

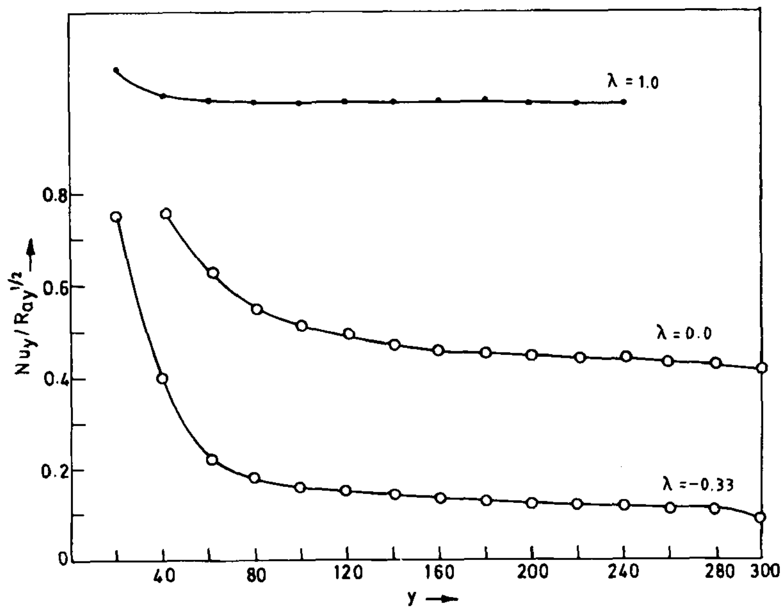


Figure 4. Variation of $Nu_y/Ra_y^{1/2}$ along the wall

for each λ. However, the present model gives a variation as shown in Figure 4. $Nu_y/Ra_y^{1/2}$ decreases with an increase in y and becomes constant at sufficiently large y. The decrease is largest in the case of $\lambda = -0{\cdot}33$ and reduces with an increase in λ. For $\lambda = 1$ the value is almost constant and coincides with the value given by Cheng and Minkowicz.

The plot of Figure 3 is almost identical to that obtained by Cheng and Minkowicz. A tabular comparison of the values is shown in Table I.

Porous medium enclosure

Next we use the method to analyse natural convection in a confined porous medium with the vertical walls at different temperatures. The system of interest is shown in Figure 1(b). A two-dimensional rectangular space of height H and horizontal dimension L is filled with a fluid-saturated porous matrix of permeability K. The same finite element model developed earlier is used here with the incorporation of appropriate boundary conditions. The top and bottom walls are insulated. The left-hand wall is taken to be warm at a temperature $\Delta T/2$, while the wall on the right is cold and held at $-\Delta T/2$. Since the region is an enclosure, ψ is to be specified as zero on all four boundaries.

Solution of the finite element equations gives the temperature distribution in the enclosure. Nu and Ra are calculated as before and a plot of Nu versus $Ra(H/L)$ is shown in Figure 5.

Table I

	$Nu_y/Ra_y^{1/2}$	
λ	Similarity solution of Cheng and Minkowicz[7]	FEM solution
−0·33	0	0·0832
−0·25	0·1621	0·1919
0	0·444	0·4173
0·25	0·6303	0·6026
0·33	0·6788	0·6569
0·5	0·7615	0·7536
0·75	0·8926	0·8811
1	1·001	0·9916

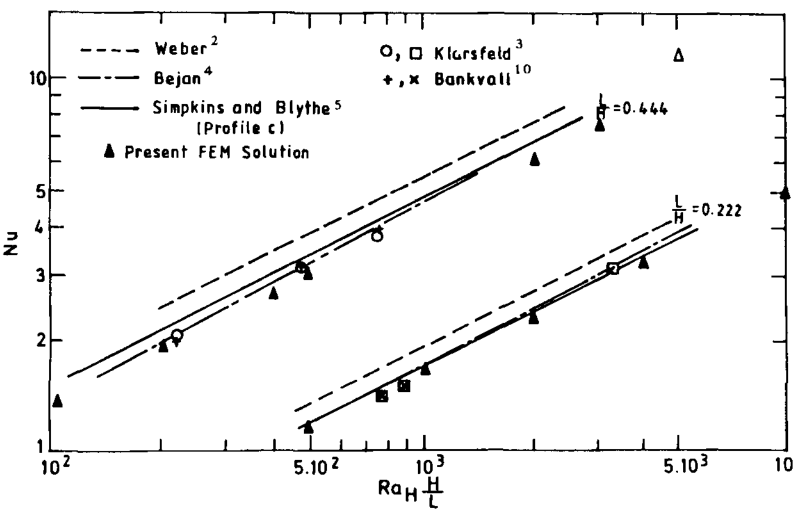


Figure 5. Comparison with experimental and other theoretical results for Nu

Two cases, with $L/H = 0{\cdot}444$ and $0{\cdot}222$, are considered. The plot also gives experimental and other theoretical results for the two cases.[2–5,10] It is seen that the agreement with the experimental results is good. The values are, however, slightly lower than the boundary layer regime values, and this is only to be expected since the present model relaxes the assumptions of boundary layer analysis. It can be observed from Figure 5 that the Nusselt number increases with the Rayleigh number for a given height and width of the porous enclosure, as expected. The nature of the variation is of the power-law type since the variation is linear on log–log scale. For a given Rayleigh number, the Nusselt number increases as the width of the enclosure increases for a given height or as the height increases for a given width.

Figure 5 also gives values of Nu in the ranges $Ra(H/L) > 3000$ and $Ra(H/L) < 200$— ranges for which results are not available in the literature.

4. CONCLUDING REMARKS

A finite element method has been developed to analyse free convection heat transfer in a porous medium. The method can be used for two-dimensional analysis of a porous body of arbitrary shape. The restriction of the boundary layer approximation is relaxed. The two cases of convection past a vertical surface embedded in the medium and convection in an enclosure are considered. The results are found to be in good agreement with those available in the literature. Results for ranges of $Ra(H/L)$ far beyond what is presently available in the literature have been obtained for the porous enclosure.

APPENDIX: NOMENCLATURE

u, v	velocities in the x- and y-directions
ψ	streamfunction
T	temperature
ρ	density
μ	viscosity
β	thermal expansion coefficient
α	thermal diffusivity
ε	porosity of the medium
K	permeability of the porous medium
k_m	thermal conductivity of the saturated medium

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